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13. ABSTRACT

A study is made of blocking pairs of polyhedra (blocking pairs of matrices) that arise in (or can be transformed into) a network flow context. For example, the blocking polyhedron of the polyhedron generated by all integral feasible flows in a capacity-constrained supply-demand network (where all the data are integral) is explicitly determined, and a simple algorithm is described for solving the associated integral packing problem. Applications of these results to k-ways in directed graphs, (0,1)-matrices with prescribed row and column sums, and to flow arborescences are described.

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BLOCKING PAIRS OF POLYHEDRA ARISING FROM NETWORK FLOWS

by

D. R. Fulkerson and David B. Weinberger

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1. Introduction. In [2] and [4] the notion of a blocking pair of polyhedra arising from combinatorial optimization problems of the maximum packing variety was introduced and studied. In this paper we characterize the appropriate polyhedra for such problems which arise in (or can be transformed into) a network flow context. We begin in Section 2 by briefly reviewing the blocking notion and some general theorems concerning blocking pairs of polyhedra. In Section 3 we consider uncapacitated supply-demand networks and note a simple decomposition property that has some important ramifications. In Section 4 we go on to the case of capacitated supply-demand networks. This section contains perhaps the most general and informative results of the paper (Theorem 4.1, Lemma 4.2, and Theorem 4.3). Finally, in Section 5 we discuss some particular combinatorial structures which fit into this context and initially motivated some of this work.

2. Blocking pairs of polyhedra. Let A be an m by n non-negative matrix and consider the polyhedron

$$\mathcal{B} = \{x \in R^n_+ \mid Ax \geq 1\}, \tag{2.1}$$

where 1 is the m-vector all of whose components are 1 and R^n_+ is the non-negative orthant of R^n. The rows of A will generally represent the combinatorial structure involved in a particular problem; for example, they might be incidence vectors of a family of subsets of an n-set. We will often refer to the polyhedron (2.1) as $\mathcal{B}(A)$ to indicate the matrix that generates it. We note that $\mathcal{B}(A)$ is n-dimensional, convex, and unbounded (except in the degenerate case where A has a zero row and hence $\mathcal{B}(A)$ is empty).

The rows of A may not all represent facets of $\mathcal{B}(A)$; that is, some of the constraints in $Ax \geq 1$ may be superfluous. Call a row vector a^i of A *inessential* if a^i dominates (is greater than or equal to) some convex combination of other row vectors of A; otherwise, call a^i *essential*. Then by the Farkas lemma on systems of linear inequalities, a row of A is superfluous in defining $\mathcal{B}(A)$ if and only if it is inessential. We call a non-negative matrix A *proper* if all of its rows are essential. (If A is a (0,1)-matrix, then A is proper if and only if it is the incidence matrix of a family of m pairwise non-comparable subsets of an n-set.)

Now let

$$\hat{\mathcal{B}} = \{x \in R^n_+ \mid x \cdot \mathcal{B} \geq 1\}; \tag{2.2}$$

that is, $\hat{\mathcal{B}}$ consists of all non-negative n-vectors such that $x \cdot b \geq 1$ for all $b \in \mathcal{B}$. We call $\hat{\mathcal{B}}$ defined by (2.2) the *blocking polyhedron* of $\mathcal{B}$. Theorem 2.1 below describes the relationship between $\mathcal{B}$ and $\hat{\mathcal{B}}$.

Theorem 2.1. Let the m by n matrix A be proper with rows $a^1,\ldots,a^m$. Let $\mathcal{B} = \{x \in R^n_+ \mid Ax \geq 1\}$ have extreme points $b^1,\ldots,b^r$, and let B be the r by n matrix with rows $b^1,\ldots,b^r$. Let $\mathcal{P} = \{x \in R^n_+ \mid Bx \geq 1\}$. Then

(i) $\hat{\mathcal{B}} = \mathcal{P}$;

(ii) B *is proper*;

(iii) *The extreme points of* $\mathcal{P}$ *are* $a^1,\ldots,a^m$;

(iv) $\hat{\mathcal{P}} = \mathcal{B}$, *and hence* $\hat{\hat{\mathcal{B}}} = \mathcal{B}$.

The matrix B of Theorem 2.1 is called the blocking matrix or blocker of A. Theorem 2.1 shows that $\mathcal{B}$ and $\hat{\mathcal{B}}$ (A and B) play symmetric roles in the relationship; together they constitute a blocking pair of polyhedra (a blocking pair of matrices). We see that for any blocking pair of polyhedra, the non-trivial facets of one and the extreme points of the other are represented by exactly the same n-vectors. In optimization contexts, one is often interested in explicitly characterizing by linear inequalities a convex polyhedron having prescribed vertices. If the matrix A with these vertices as its rows is proper, then the blocking polyhedron of $\mathcal{B}(A)$ yields one such characterization (not of the convex hull of the rows of A, but rather of the vector sum of this convex hull with the non-negative orthant).

Now let A be as in Theorem 2.1 and consider the following maximum packing problem:

$$(2.3)\qquad \begin{aligned} yA &\leq w \\ y &\geq 0 \\ \max\ &1\cdot y, \end{aligned}$$

where $w \in R^n_+$ and 0 and 1 are the m-vectors all of whose components are 0 and 1 respectively. Let B be an r by n non-negative matrix having rows $b^1,\ldots,b^r$. Say that the max-min equality holds for the (ordered) pair A,B if and only if, for every $w \in R^n_+$, the packing problem (2.3) has an optimal solution vector y such that

$$(2.4)\qquad 1\cdot y = \min_j b^j\cdot w.$$

Say that the <u>min-min inequality</u> holds for the (unordered) pair A,B if and only if, for every $\ell \in R^n_+$ and $w \in R^n_+$, we have

$$(2.5) \qquad (\min_i a^i \cdot \ell)(\min_j b^j \cdot w) \leq \ell \cdot w.$$

Theorem 2.2 below shows that the blocking relation is essentially equivalent to these notions.

Theorem 2.2. (i) Let A and B be a blocking pair of matrices. Then the max-min equality holds for both ordered pairs A,B and B,A, and the min-min inequality holds for the unordered pair A,B.

(ii) Let A and B be proper matrices. If the max-min equality holds for the pair A,B (in either order), then A and B are a blocking pair of matrices.

(iii) Let A and B be proper matrices whose rows satisfy $a^i \cdot b^j \geq 1$ for all i and j. If the min-min inequality holds for the pair A,B, then A and B are a blocking pair of matrices.

Note that the max-min equality and the min-min inequality hold for any pair of matrices A,B that generate a blocking pair of polyhedra, i.e. adding inessential rows to either A or B affects neither the max-min equality nor the min-min inequality. The only reason for restricting the matrices in the theorem to be proper is so that we can in fact claim that A and B are a blocking pair of matrices (and hence be assured that each row of one occurs as an extreme point of the polyhedron $\mathcal{B}$ generated by the other).

Max-min type results are common in combinatorial optimization problems, and, through Theorem 2.2, the blocking theory (and its parallel counterpart,

the anti-blocking theory [3,4], which is aimed at minimum covering problems) can be viewed as an attempt to identify and further understand the common ground on which such results rest.

There is one more theorem concerning blocking pairs that will be needed in what follows. Let A be a non-negative matrix. Contracting coordinate j (column j) in A means dropping the j^{th} column from the matrix A. Deleting coordinate j (column j) means dropping the j^{th} column from A and dropping all rows from A that had a positive entry in this column. Theorem 2.3 below shows that these operations are dual to each other in the blocking context.

Theorem 2.3. *Let* A *and* B *be non-negative matrices such that* $\mathcal{B}(A)$ *and* $\mathcal{B}(B)$ *are a blocking pair of polyhedra. If we contract the* j^{th} *coordinate of* A, *leaving* A', *and delete the* j^{th} *coordinate of* B, *leaving* B', *then* $\mathcal{B}(A')$ *and* $\mathcal{B}(B')$ *are a blocking pair of polyhedra.*

We also note that in any sequence of contractions and deletions on A, say, the order in which these operations are performed is immaterial.

3. Uncapacitated supply-demand networks. Let $[N,\mathcal{A}]$ be an arbitrary network with node-set N and arc-set $\mathcal{A}$. (All networks considered in this paper will be directed--elements of $\mathcal{A}$ are ordered pairs of elements of N.) Let some non-empty subset S of N be considered "source" nodes and some non-empty subset T of N be considered "sink" nodes, where $S \cap T = \emptyset$. With each node $x \in S$ we associate a non-negative number $a(x)$, the "supply" at x, and with each node $x \in T$ we associate a non-negative number $b(x)$, the "demand" at x. We assume, without any real loss of generality, that

$$\sum_{x \in S} a(x) = \sum_{x \in T} b(x). \tag{3.1}$$

A <u>feasible flow</u> for this system is a function $f: A \to R_+$ such that

$$f(x,N) - f(N,x) = a(x) \quad \text{for all} \quad x \in S, \tag{3.2}$$

$$f(N,x) - f(x,N) = b(x) \quad \text{for all} \quad x \in T, \tag{3.3}$$

$$f(x,N) - f(N,x) = 0 \quad \text{for all} \quad x \notin S \cup T. \tag{3.4}$$

Here R_+ denotes the non-negative reals and $f(x,N)$, for example, denotes the sum

$$\sum_{\{y \in N | (x,y) \in A\}} f(x,y).$$

(Later on, when we add a capacity function $c: A \to R_+$ to the network, a feasible flow f will also have to satisfy $f(x,y) \leq c(x,y)$ for all $(x,y) \in A$.) Throughout, we will use the notation, for arbitrary $X \subseteq N$, $Y \subseteq N$,

$$(X,Y) = \{(x,y) \in A | x \in X, \ y \in Y\}, \tag{3.5}$$

and if g is any real-valued function defined on the arcs,

$$g(X,Y) = \sum_{(x,y) \in (X,Y)} g(x,y). \tag{3.6}$$

Similarly, if h is any real-valued function defined on the nodes, and if $X \subseteq N$, we write

$$h(X) = \sum_{x \in X} h(x). \tag{3.7}$$

Let us assume all data (i.e. supplies and demands) are integers and consider the finite list of all integral feasible flows. Let A be the matrix whose columns are indexed by the arcs of the network and whose rows are indexed by the integral feasible flows, with entry a_{ij} representing the amount flowing in arc j in the i^{th} flow. Note that A may not be proper; also note that A may have no rows. Our first goal is to determine the blocking polyhedron of $\mathcal{B}(A)$.

We begin by establishing a simple but important decomposition lemma. To simplify the statement of this lemma, let N represent the supply-demand system given by the network $[N,A]$ and supply and demand functions a and b, respectively. Then we let N_p denote the supply-demand system given by the same network $[N,A]$ but with supplies and demands multiplied throughout by $p \in R_+$. Let Z_+ denote the non-negative integers.

Lemma 3.1. Every integral feasible flow for N_k, $k \in Z_+$, $k \geq 1$, can be decomposed into k integral flows, each feasible for N.

Proof. We proceed by induction on k. The lemma is trivial for $k = 1$. Now let f be an integral feasible flow for N_k, $k > 1$. We wish to extract an integral subflow from f which is feasible for N and leave behind a flow that is feasible for N_{k-1}. By the induction hypothesis this latter flow will decompose into $k-1$ integral flows, each feasible for N, and we shall be done. But it is clear that the removal from f of any integral subflow feasible for N will leave an integral flow feasible for N_{k-1}. So we need only show that f contains an integral subflow feasible for N.

Consider f to be a capacity function imposed on the system N. If this system is feasible (i.e., if there exists a feasible flow for it), then there will exist an integral feasible flow by well-known integrality properties of network flows [1], and such a flow will clearly be an appropriate subflow of f. By the supply-demand theorem [6; 1,Th. II.1.1], the system N with capacity function f is feasible if and only if

$$(3.8) \qquad b(T \cap \overline{X}) - a(S \cap \overline{X}) \leq f(X,\overline{X})$$

for all $X \subseteq N$, where $\overline{X} = N - X$. Now since f is a feasible flow for N_k, we have

$$(3.9) \qquad k(b(T \cap \overline{X}) - a(S \cap \overline{X})) = f(X,\overline{X}) - f(\overline{X},X)$$

for all $X \subseteq N$, i.e., the net demand over a subset $\overline{X}$ of nodes is equal to the net flow into those nodes. Thus (3.9), the non-negativity of f, and the assumption $k > 1$, imply (3.8). Hence there does exist an appropriate subflow of f, proving Lemma 3.1.

We now use Lemma 3.1 and Theorem 2.2 to establish the blocking polyhedron of $\mathcal{B}(A)$, where A is the matrix of integral feasible flows.

Theorem 3.2. Let A be the matrix of integral feasible flows in an uncapacitated supply-demand network $[N,A]$ with integral-valued supply and demand functions, a and b, respectively. Then the blocking polyhedron of $\mathcal{B}(A)$ is described by the constraints

$$(3.10) \qquad \xi_e \geq 0 \text{ for all } e \in A,$$

$$\sum_{e \in (X,\overline{X})} \xi_e \geq b(\overline{X}) - a(\overline{X}), \tag{3.11}$$

<u>for all</u> $X \subseteq N$ <u>such that</u> $b(\overline{X}) - a(\overline{X}) \geq 1$.

In other words, if we let B be the matrix whose columns represent the arcs of the network, having a row for each $X \subseteq N$ such that $b(\overline{X}) - a(\overline{X}) \geq 1$, with entry $1/(b(\overline{X}) - a(\overline{X}))$ in each column representing an arc of $(X,\overline{X})$ and zero entries elsewhere, then the essential rows of A and B form a blocking pair of matrices.

<u>Proof</u>. Let $w \in R^n_+$ and consider the packing problem

$$\begin{aligned} &yA \leq w \\ &y \geq 0 \\ &\max\ 1 \cdot y, \end{aligned} \tag{3.12}$$

i.e. find the maximum weight packing of rows of A into the arc-weight vector w. We note that a packing of total weight r, $r \in R_+$, exists if and only if the system N_r with capacity function w on the arcs is feasible. Necessity is clear and we demonstrate sufficiency now. If $r = 0$, sufficiency is clear, so assume $r > 0$. Let f be a feasible flow to N_r with arc capacities given by the components of w. We must show that f can be written as a positive linear combination of integral feasible flows for N. Consider the flow f_r defined by $f_r(x,y) = f(x,y)/r$ for all $(x,y) \in A$. The flow f_r is feasible for N and hence can be written as a convex combination of integral feasible flows for N, since, as is well known, the feasible flows in a network with integral data form a convex polyhedron with integral extreme points. Multiplying each of

the non-zero coefficients in this convex combination by r gives us a positive linear combination of integral feasible flows (rows of A) equalling f, and hence a feasible packing with total weight r. So sufficiency is established.

Now, by the supply-demand theorem, N_r with capacities given by the components of w is feasible if and only if

$$r(b(\overline{X} \cap T) - a(\overline{X} \cap S)) = r(b(\overline{X}) - a(\overline{X})) \leq w(X,\overline{X})$$

for all $X \subseteq N$. Hence the maximum feasible r, say r^*, is given by

$$r^* = \min_{X \subseteq N} \frac{w(X,\overline{X})}{b(\overline{X}) - a(\overline{X})}, \tag{3.13}$$

where the minimum is taken over all $X \subseteq N$ such that the denominator in (3.13) is positive. Now consider the matrix B described above. By *Theorem 2.2*(ii), we have just established that the essential rows of A and the essential rows of B form a blocking pair of matrices. Hence the polyhedron given by (3.10) and (3.11) is the blocking polyhedron of $\mathcal{B}(A)$.

Note that, as a by-product of the proof of Theorem 3.2, we get the following.

Theorem 3.3. Let A be the matrix of integral feasible flows for an uncapacitated supply-demand system N with integral data. Given $w \in Z_+^n$ (a non-negative integral weight function on the arcs of the network), let r^* be the total weight of an optimal packing, i.e.

(3.14)
$$yA \leq w$$
$$y \geq 0$$
$$\max 1 \cdot y = r^*,$$

and let z^* be the total weight of an optimal integral packing, i.e.

(3.15)
$$yA \leq w$$
$$y \geq 0, \quad y \text{ integral}$$
$$\max 1 \cdot y = z^*.$$

Then $z^* = [r^*]$, where brackets denote the biggest integer function.

Proof. That r^* is optimal in (3.14) implies that N_{r^*} with capacity function w is feasible. This implies that $N_{[r^*]}$ with capacity function w is feasible, and hence there is an integral feasible flow to $N_{[r^*]}$ with capacity function w. The case $[r^*] = 0$ gives no difficulty, so assume $[r^*] \geq 1$. Then by Lemma 3.1, this flow can be decomposed into $[r^*]$ integral flows for N, i.e. there is an integral non-negative packing y in (3.15) with component sum equal to $[r^*]$. Hence $[r^*]$ is achievable in (3.15), and thus $z^* \geq [r^*]$. But clearly $z^* \leq r^*$, and consequently the integer $z^* = [r^*]$.

While Theorems 3.2 and 3.3 are interesting in their own right, their main function is to lead us to more general theorems of this kind concerning capacitated supply-demand networks. We proceed to this situation in the next section.

4. Capacitated supply-demand networks. Our network will now be assumed to have a capacity function $c: A \to R_+$, and a feasible flow f will satisfy the additional property that $f(x,y) \leq c(x,y)$ for all $(x,y) \in A$. Given such a supply-demand system (we still assume $a(S) = b(T)$) with integral data (supplies, demands, and arc capacities), we let A be the matrix of all integral feasible flows as before and again ask for the blocking polyhedron of $\mathcal{B}(A)$.

We make use of a known technique [9; 1,p.129] to reduce our capacitated supply-demand problem to an uncapacitated one. Given the network $[N,A]$ above, we construct a new (uncapacitated) bipartite supply-demand network $[N',A']$. For every arc in the original network, we have a source node in the new one, labelled with the ordered pair (x,y), for $(x,y) \in A$. For every node in the original network, we have a sink node in the new one, labelled x, where $x \in N$. Source (x,y) is joined by arcs from (x,y) to sinks x and y in the new network, and these are the only arcs in A'. Source (x,y) has supply $c(x,y)$ and sink x has a demand of $c(x,N) - a(x)$, $c(x,N) + b(x)$, or $c(x,N)$, according as node x was a source, sink, or neither in $[N,A]$. Note that the sum of the supplies in this new network still equals the sum of the demands. (Figure 4.1 below shows a capacitated supply-demand network and the uncapacitated one derived from it.)

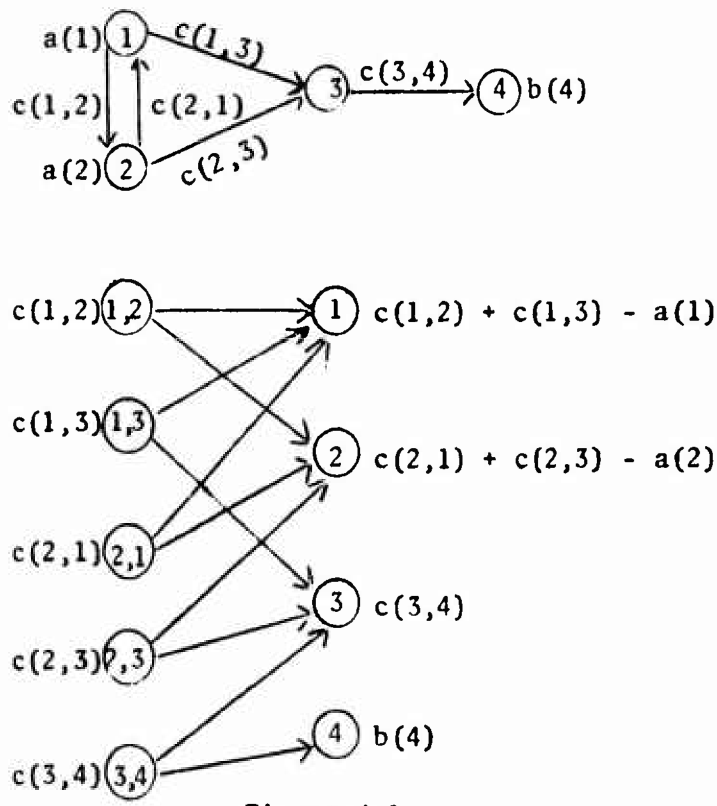


Figure 4.1

The relationship between the two networks is as follows. For every feasible flow f' in the new network, the flow f in the original network defined by $f(x,y) = f'((x,y),y)$ is feasible, and conversely, for every feasible flow f in the original network, the flow f' in the new network defined by $f'((x,y),y) = f(x,y)$ and $f'((x,y),x) = c(x,y) - f(x,y)$ is feasible.

We shall use this transformation to determine the blocking polyhedron of $\mathcal{B}(A)$.

Theorem 4.1. Let A *be the* m *by* n *matrix of integral feasible flows in a capacitated supply-demand network* $[N,A]$ *with integral-valued supply, demand, and capacity functions* a, b, *and* c, *respectively. Then the blocking polyhedron of* $\mathcal{B}(A)$ *is given by the constraints*

(4.1) $$\xi_e \geq 0 \quad \underline{\text{for all}} \quad e \in A,$$

(4.2) $$\sum_{e \in E \subseteq (X,\overline{X})} \xi_e \geq b(\overline{X}) - a(\overline{X}) - c((X,\overline{X}) - E),$$

<u>for all</u> $X \subseteq N$ <u>and all</u> $E \subseteq (X,\overline{X})$ <u>such that the right-hand side of</u> (4.2) <u>is positive</u>.

In other words, if we let B be the matrix whose columns correspond to arcs of A, having a row for each $X \subseteq N$ and each $E \subseteq (X,\overline{X})$ such that the right-hand side of (4.2) is positive, with entry $1/(b(\overline{X}) - a(\overline{X}) - c((X,\overline{X}) - E))$ in each column corresponding to an arc of E and zero entries elsewhere, then the essential rows of A and the essential rows of B are a blocking pair of matrices.

<u>Proof</u>. Consider the uncapacitated network $[N',A']$ described above and let A' be the m by $2n$ matrix of integral feasible flows for it. By Theorem 3.2, the blocking polyhedron of $\mathcal{B}(A')$ is given by

(4.3) $$\xi_e \geq 0 \quad \text{for all} \quad e \in A',$$

(4.4) $$\sum_{e \in (X,\overline{X})} \xi_e \geq \text{demand}(\overline{X}) \qquad y(\overline{X})$$

for all $X \subseteq N'$ such that the right-hand side of (4.4) is positive. Letting F and W be the source and sink nodes, respectively, contained in X, we can write (4.4), in terms of the original data, as

(4.5) $$\sum_{e \in (F,\overline{W})} \xi_e \geq c(\overline{W},N) + b(\overline{W}) - a(\overline{W}) - c(\overline{F}),$$

(here $\overline{W} = N - W$, $\overline{F} = A - F$) for all $F \subseteq A$ and $W \subseteq N$, where $(F,\overline{W})$ is the collection of arcs of A' which are of the form $((x,y),z)$ with $(x,y) \in F$ and $z \in \overline{W}$, and of course, the right-hand side of (4.5) is positive.

Now the matrix A is gotten from the matrix A' by contracting the columns corresponding to arcs of the form $((x,y),x)$. Hence, by Theorem 2.3, the blocking polyhedron of $\mathcal{B}(A)$ will be gotten by dropping from (4.5) all constraints that involve such arcs, i.e. by deleting the appropriate columns from the blocking matrix of A'. Thus our desired polyhedron is given by (4.1) and by the constraints

$$\sum_{e\in(F,\overline{W})} \xi_e \geq c(\overline{W},N) + b(\overline{W}) - a(\overline{W}) - c(\overline{F}) \tag{4.6}$$

for all $F \subseteq A$ and $W \subseteq N$ such that F contains no arc (of A) whose tail is in $\overline{W}$. Thus for given $W \subseteq N$, we only consider constraints where $F \subseteq (W,N) \subseteq A$. Now any constraint with $F \not\supseteq (W,W)$ is clearly inessential because adding the missing edges of (W,W) to F increases the right-hand side of the constraint without changing the left-hand side. Hence for given $W \subseteq N$, we need only consider constraints where F is of the form $(W,W) \cup E$ with $E \subseteq (W,\overline{W})$. Hence our desired polyhedron is given by (4.1) and

$$\sum_{e\in E\subseteq(W,\overline{W})} \xi_e \geq c(\overline{W},N) + b(\overline{W}) - a(\overline{W}) - c(A - ((W,W) \cup E)) \tag{4.7}$$

The right-hand side of (4.7) can be written as $c(\overline{W},N) + b(\overline{W}) - a(\overline{W}) - c(\overline{W},N) - c(W,\overline{W}) + c(E)$, whence replacing W by X gives the desired result.

Note that the transformation technique used above allows us to generalize Lemma 3.1 to the case of capacitated supply-demand networks. Let N_p, for $p \in R_+$, refer now to the supply-demand network where all supplies, demands, and capacities have been multiplied throughout by p.

Lemma 4.2. Let N be a capacitated supply-demand system with integral data. Then every integral feasible flow for N_k, where k is a positive integer, can be decomposed into k integral flows, each feasible for N.

Proof. Use the transformation depicted in Figure 4.1 to reduce the problem to an uncapacitated one and apply Lemma 3.1.

Theorem 3.3 also generalizes to the capacitated case:

Theorem 4.3. Let A be the m by n matrix of integral feasible flows for a capacitated supply-demand network with integral data. Given $w \in Z_+^n$ (a non-negative integral weight function on the arcs of the network), let r^* and z^* be weights of optimal solutions to the packing problems (3.14) and (3.15), respectively. Then $z^* = \lceil r^* \rceil$.

Proof. Consider the transformed network as above and apply Theorem 3.3 to that network with weight function $w(x,y)$ on arcs $((x,y),y)$ and ∞ on arcs $((x,y),x)$.

In the next section we shall apply these results to some part combinatorial problems, but first we conclude this section with an example illustrating Theorem 4.1, Theorem 4.3, and with some remarks.

Example. Let the capacitated network be that shown in Figure 4.2 below:

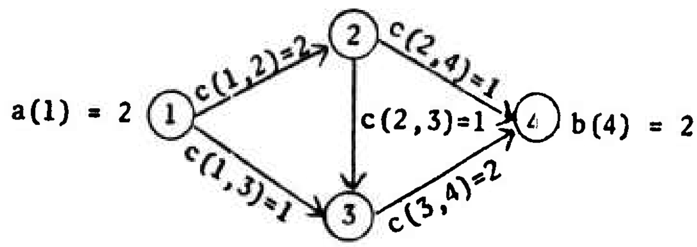


Figure 4.2

The matrix of integral feasible flows is

$$A = \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 \end{bmatrix}$$

where the columns are indexed from left to right by arcs (1,2), (1,3), (2,3), (2,4), (3,4). The blocking matrix B of A is

$$B = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $w = (7,2,2,7,7)$. To solve the integer packing problem for A and w, we can proceed as follows. Successively multiply the supplies and demands by $k = 0,1,2,\ldots,$ using as capacities in the network $c_k(x,y) = \min(kc(x,y), w(x,y))$ for the k^{th} problem. The largest k for

which the problem is feasible is the answer z^* of Theorem 4.3. In the example, $z^* = 4$, and a maximum packing vector for the rows of A is given by $y = (2,2,0)$. (Note that $\min_j b^j \cdot w = 4$, the minimum being achieved at the 4^{th} row of B.) But it is not true, as one might think from a casual reading of the proof of Theorem 4.3 (which ultimately rests, through the transformation to an uncapacitated network, on the proof of Lemma 3.1) that the best integer packing vector y can be found by attempting to decompose the final flow by first extracting an arbitrary integral feasible subflow corresponding to $k = 1$, and so on. For instance, in the example, there is no optimal integer packing for A that assigns its last row a positive weight. One way to find the best integral packing vector y is to pass to the transformed uncapacitated bipartite network, its corresponding final flow, and then use the proof of Lemma 3.1. The interested reader may wish to do this in the example.

The thrust of our remarks, as illustrated by the example, is two-fold. Firstly, Theorem 4.3 is more subtle than one might think. (We feel that it is, indeed, a surprising result.) Secondly, there is a reasonably efficient algorithm for solving the integer packing problem described in Theorem 4.3.

We have said nothing about the max-min equality for the ordered pair B,A, but it may be worth noting that finding a row of A that achieves $\min_i a^i \cdot w$ is equivalent to solving a minimum-cost network flow problem [1], where $w(x,y)$ is now interpreted as the cost per unit of flow in arc (x,y).

5. <u>Some special cases</u>. In this section we discuss some particular classes of integral packing problems, each of which fits into the general context of the preceding section. In each case we start with a non-negative

integral matrix A whose rows correspond to integral feasible flows in a capacitated supply-demand network with integral data, describe the blocking polyhedron of $\mathcal{B}(A)$, and briefly discuss the integral packing problem for A and a non-negative integral weight vector w having a component for each column of A.

Example 5.1 (k - ways in directed graphs). Suppose we have a single source s and single sink t in our flow network with $a(s) = b(t) = k$, where k is a positive integer, and assume all arc capacities are 1. Then an integral feasible flow (if one exists) can be decomposed into a collection of k arc-disjoint directed paths from s to t, plus possibly some arc-disjoint directed circuits. We may throw away the circuits, if any, in such a decomposition, since retaining them would clearly yield inessential rows in our (0,1)-matrix A. Each row of A may then be viewed as the incidence vector of a k - way from s to t. (Some of these may still be inessential.) The constraints (4.2) of Theorem 4.1 simplify to

$$\sum_{e \in E \subseteq (X,\bar{X})} \xi_e \geq k - |(X,\bar{X}) - E|, \tag{5.1}$$

where $(X,\bar{X})$ is a cut separating $s \in X$ from $t \in \bar{X}$, $|Z|$ denotes the cardinality of set Z, and we have a constraint (5.1) for each subset E of every cut $(X,\bar{X})$ separating s from t such that the right-hand side of (5.1) is positive. (Notice that if $k = 1$, so that A is the incidence matrix of s to t simple directed paths, then we must take $E = (X,\bar{X})$, and the blocking matrix B of A is a (0,1)-matrix, the incidence matrix of all set-wise minimal cuts separating s from t [2,4].)

The integral packing problem for A and a non-negative integral weight vector w can be solved by the general procedure described in Section 4.

The optimal integral packing vector y has component sum equal to

$$\min_{E \subseteq (X,\bar{X})} \left[\frac{w(E)}{k - |(X,\bar{X}) - E|} \right], \tag{5.2}$$

where the minimum is taken over all subsets E of s,t cuts $(X,\bar{X})$ such that the denominator in (5.2) is positive.

Analogous results hold for undirected graphs.

<u>Example 5.2</u> (<u>Zero-one matrices with prescribed row and column sums</u>). Let $a_1,\ldots,a_m$ and $b_1,\ldots,b_n$ be positive integers and consider the class $\mathfrak{A}(a,b)$ of all m by n (0,1)-matrices having row sums $a_1,\ldots,a_m$ and column sums $b_1,\ldots,b_n$ respectively [1,6,8]. There is a simple criterion due independently to Ryser (see [8]) and Gale [6] in order that the class $\mathfrak{A}(a,b)$ be non-empty; this criterion is in terms of the majorization concept for the vectors $a = (a_1,\ldots,a_m)$ and $b = (b_1,\ldots,b_n)$. There is also a simple algorithm for constructing a matrix in the class, if one exists, or ascertaining that the class is empty (see [1,6,8] for a full discussion of these matters).

Now let A be the matrix having mn columns, corresponding to the cells of matrices in $\mathfrak{A}(a,b)$; each row of A is a (0,1)-vector corresponding to a member of $\mathfrak{A}(a,b)$. In other words, A is the incidence matrix of the class of all members of $\mathfrak{A}(a,b)$. (The matrix A is proper; it *may* have no rows.)

Since members of $\mathfrak{A}(a,b)$ correspond precisely to feasible integral flows in a complete bipartite graph with m *source* nodes, having supplies $a_1,\ldots,a_m$, in one node-part, and n sink nodes, having demands $b_1,\ldots,b_n$, in the other node-part, where all arcs lead from sources to sinks and

have capacity 1, the incidence matrix A described above fits into the context of Section 4. Theorem 4.1 then implies:

Theorem 5.1. Let A be the (0,1)-matrix described above whose rows represent the members of the class $\mathcal{O}\!\mathcal{I}(a,b)$. Then the blocking polyhedron of $\mathcal{B}(A)$ is given by the constraints

$$\xi_{ij} \geq 0 \quad \text{for} \quad i = 1,\ldots,m \quad \text{and} \quad j = 1,\ldots,n, \tag{5.3}$$

$$\sum_{(i,j)\in E\subseteq R\times C} \xi_{ij} \geq b(C) - a(\bar{R}) - |R||C| + |E|, \tag{5.4}$$

for all $R \subseteq \{1,\ldots,m\}$, $C \subseteq \{1,\ldots,n\}$, and all $E \subseteq R \times C$ such that the right-hand side of (5.4) is positive.

Here we have doubly subscripted the variables to correspond to the cells of our m by n matrices. R is a set of row indices ($\bar{R}$ is the complementary set), C is a set of column indices, and E is a subset of the cells in rows R and columns C.

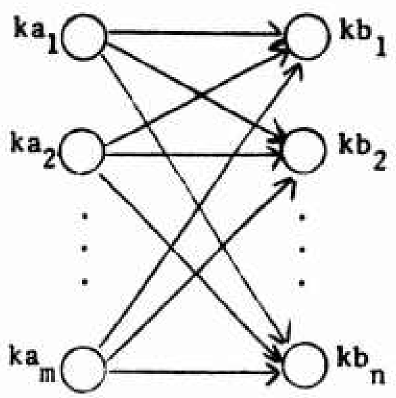
Theorem 4.3 then implies:

Theorem 5.2. Let $w \in Z_+^{mn}$ be a weight function on the cells of an m by n matrix (i.e. w is a non-negative integral m by n matrix), and let A be the incidence matrix of Theorem 5.1. Then the maximum value z^* in the integral packing problem (3.15) is given by

$$z^* = \min_{E\subseteq R\times C} \left[\frac{w(E)}{b(C) - a(\bar{R}) - |R||C| + |E|} \right] \tag{5.5}$$

where the minimum in (5.5) is taken over all choices of R, C, and $E \subseteq R \times C$ such that the denominator is positive.

In other words, Theorem 5.2 describes the maximum number of members of $\mathcal{M}(a,b)$ that can be packed into an m by n non-negative integral matrix. We also note that to determine z^*, we need not compute the right-hand side of (5.5), but can simply solve the sequence of network-flow problems pictured in Figure 5.1 below for $k = 0,1,2,\ldots$.



(arc capacities $\min(k,w_{ij})$)

Figure 5.1

The largest feasible k will be z^*.

Theorem 5.2 allows us to treat certain feasibility problems for 3-dimensional (0,1)-arrays in a 2-dimensional context. Such problems are in general notoriously difficult. Perhaps the most comprehensive work on multi-dimensional arrays that is related to the material of this paper is due to Jurkat and Ryser [7], who remark that a general existence theorem for 3-dimensional (0,1)-arrays with prescribed line sums is unknown. The results of this section do not of course furnish such a theorem, but they do give information on the following special case. Let $a_1,\ldots,a_m$ and $b_1,\ldots,b_n$ be row and column sums at each level of an m by n by p (0,1)-array. Then Theorem 5.2 implies the following:

<u>Theorem 5.3.</u> <u>The constraints</u>

(5.6) $$\sum_{j=1}^{n} x_{ijk} = a_i \quad \underline{\text{for}} \quad 1 \le i \le m, \quad 1 \le k \le p,$$

(5.7) $$\sum_{i=1}^{m} x_{ijk} = b_j \quad \underline{\text{for}} \quad 1 \le j \le n, \quad 1 \le k \le p,$$

(5.8) $$\sum_{k=1}^{p} x_{ijk} \le w_{ij} \quad \underline{\text{for}} \quad 1 \le i \le m, \quad 1 \le j \le p,$$

(5.9) $$x_{ijk} = 0 \text{ or } 1 \quad \underline{\text{for all}} \quad i, j, k,$$

<u>are feasible if and only if</u> $p \le z^*$ <u>defined by</u> (5.5).

In connection with Theorem 5.3, it should be remarked that we do not need to include the Gale-Ryser criterion that $\mathfrak{A}(a,b)$ be non-empty in the feasibility condition of Theorem 5.3. For if $\mathfrak{A}(a,b) = \emptyset$, then z^* defined by (5.5) can be shown to be zero by appropriately selecting R, C, and $E = \emptyset$.

To illustrate the integer packing problem for (0,1)-matrices having prescribed row and column sums, the reader may wish to try his hand on the following numerical example. Let $a = b = (2,2,2,2)$ and let

$$w = \begin{bmatrix} 0 & 9 & 8 & 7 \\ 9 & 3 & 2 & 9 \\ 7 & 3 & 3 & 3 \\ 7 & 3 & 3 & 3 \end{bmatrix}.$$

Choosing $R = \{2,3,4\}$, $C = \{2,3,4\}$, and $(R \times C) - E = \{(2,4)\}$ shows that $z^* \le [\frac{23}{3}] = 7$. Does $z^* = 7$?

Example 5.3 (Flow arborescences). Given a directed graph G and a particular node s of the graph, a spanning arborescence rooted at s is a set of arcs of G that forms a spanning tree of the underlying undirected graph such that (i) each node of G other than s has just one arc of the arborescence directed into it, and (ii) no arc of the arborescence is directed into s. The blocking polyhedron of $\mathcal{B}(A)$, where A is the incidence matrix of all spanning arborescences rooted at s, has been explicitly determined in [5]. Here we modify the notion of spanning arborescence in the following way: a flow arborescence rooted at s is a spanning arborescence rooted at s with each arc carrying a "flow" equal to the number of nodes that follow it in the arborescence, that is, if we think of $a(s) = p - 1$, where G has p nodes, and $b(x) = 1$ for $x \neq s$, then a flow arborescence rooted at s is an integral feasible supply-demand flow which has a spanning arborescence rooted at s as its support. It can be shown (we omit the proof) that, in this situation, if we let A' denote the matrix of all integral feasible supply-demand flows, then a row of A' that is not a flow arborescence rooted at s is inessential.

Theorem 5.4. Let A denote the m by n integral matrix of all flow arborescences rooted at s in a directed graph G having n arcs. Then the blocking polyhedron of $\mathcal{B}(A)$ is given by

$$\xi_e \geq 0, \quad \text{for all arcs } e, \tag{5.10}$$

$$\sum_{e \in (X,\bar{X})} \xi_e \geq |\bar{X}|, \quad \text{for all subsets } X \text{ of} \tag{5.11}$$

nodes such that $s \in X$.

<u>Proof.</u> This follows from Theorem 4.1 by taking $c = \infty$ on all arcs, $a(s) = p - 1$, $b(x) = 1$ for $x \neq s$ (where G has p nodes), since $b(\bar{X}) = |\bar{X}|$ or $|\bar{X}| - 1$ and $a(\bar{X}) = 0$ or $p - 1$, according as $s \in X$ or $s \in \bar{X}$.

We conclude with a word of caution on the packing problem for the matrix A of Theorem 5.4. We might be tempted to infer that the maximum integral packing of flow arborescences into a vector $w \in Z_+^n$ is given by

$$(5.12) \qquad z^* = \min_{\{X \subseteq N \mid s \in X\}} \left[\frac{w(X,\bar{X})}{|\bar{X}|} \right].$$

(It is true that the maximum real (or rational) packing is given by (5.12) without the brackets.) But (5.12) is incorrect for the best integer packing; we need all the integral feasible flows (i.e. the matrix A'), not just the essential ones, for (5.12) to be valid. For instance, consider the directed graph G with weights on the arcs shown in Figure 5.2 below.

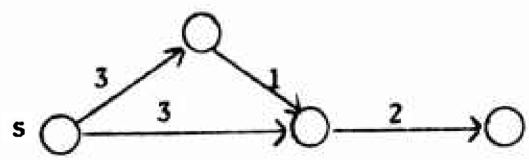


Figure 5.2

There are two flow arborescences rooted at s:

$$A = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

where $w = (3,3,1,2)$. Taking $y = (1/2,\ 3/2)$ gives a packing of weight 2,

but the best integer packing vector y has component sum $1 \neq [2]$. To get an integral packing of weight 2, we need to consider

$$A' = \begin{bmatrix} 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix}$$

whose last row is inessential, and take $y = (0,1,1)$.

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